

# Real Options: Mathematical Preliminaries

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- 1 Discrete and Continuous Random Walk
- 2 Stochastic Differential Equations: Diffusion
- 3 Models for Price Process
- 4 Pricing Options
  - Risk Neutral Probability
  - PDE Method for Black-Scholes
- 5 Stochastic Controlled Processes
- 6 Optimal Stopping Time

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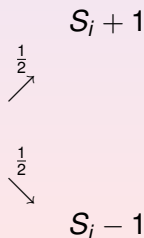
# Discrete Random Walk

- ①  $X_i$  iid random variables  $\pm 1$  with probability  $\frac{1}{2}$

$$S_0 = x$$

$$S_{i+1} = S_i + X_i$$

- ② One Step Random Walk Dynamic  $S_i$



# Properties

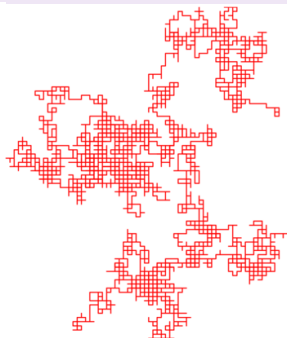
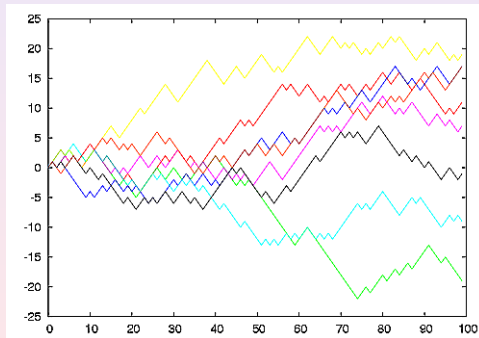
- 1 Independent Increments  $X_i$
- 2 Markov Property (Knowing  $S_i$ , the path after  $i$  is independent of the path until  $i$ )

$$\mathbb{P} [S_j = k | S_1, \dots, S_{j-1}] = \mathbb{P} [S_j = k | S_{j-1}]$$

- 3  $\mathbb{E} [S_j] = 0$  and  $\mathbb{E} [S_j^2] = j$
- 4 Martingale:

$$\mathbb{E} [S_j | S_i] = S_i; \quad j \geq i$$

# Visualization



# Brownian Motion

- 1  $W_t^N := \frac{1}{\sqrt{N}} \sum_{i=1}^{\lfloor Nt \rfloor} S_i$  converges to some stochastic process  $W_t$
- 2 Each  $W_t^N$  inherit the same properties as  $S_t$  and so the limit  $W_t$
- 3 By central limit theorem:

$$W_t \sim \text{gaussian}(0, t)$$

# Properties

- 1 Independent Increments  $\Delta_h W_t = W_{t+h} - W_t$
- 2 Markov Property (Knowing  $W_t$ , the path after  $t$  is independent of the path until  $t$ )

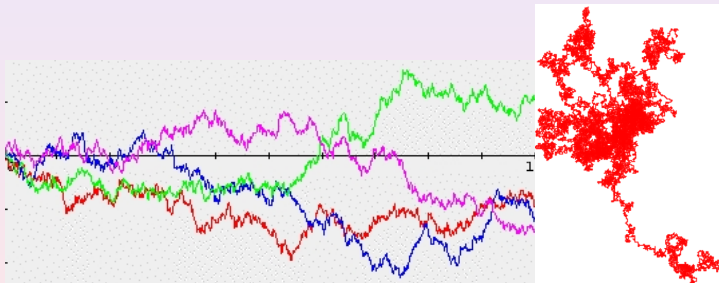
$$\mathbb{P}[W_s \in \mathcal{A} | W_u; 0 \leq u \leq t] = \mathbb{P}[W_s \in \mathcal{A} | W_t]$$

- 3  $\mathbb{E}[W_t] = 0$  and  $\mathbb{E}[W_t^2] = t$
- 4 Martingale:

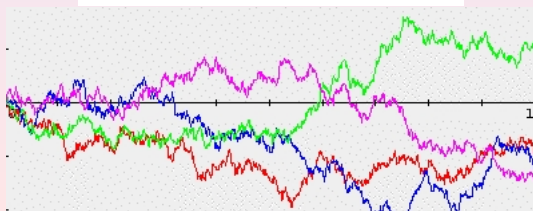
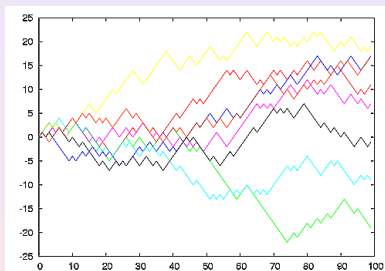
$$\mathbb{E}[W_s | W_t] = W_t; \quad s \geq t$$

- 5 **Nor** regular **neither** of bounded variation **but** continuous paths  
so  $dW_t$  do not have classical sense

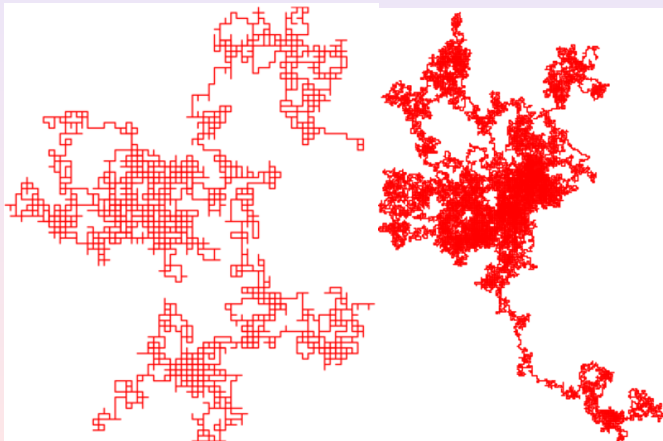
# Visualization



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# SDE

- 1  $W$  Brownian Motion
- 2  $dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$  Problem: Irregular Pathes
- 3  $X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$  Problem: Of Infinite Variations
- 4 Itô Integral: Based on Starting Point of the interval
- 5  $\sum_{t_i \leq t} \sigma(t_i, X_i)(W_{t_{i+1}} - W_{t_i})$  converge to some stochastic process

$$\int_0^t \sigma(s, X_s)dW_s$$

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$$\int_0^t \sigma(s, X_s)dW_s$$

# Properties

- 1 Itô Integral is Martingale
- 2  $\mathbb{E} \left[ \left( \int_0^t \sigma(s, X_s) dW_s \right)^2 \right] = \mathbb{E} \left[ \int_0^t \sigma(s, X_s)^2 ds \right]$
- 3 Itô formula (Chain Rule):  $V$  two times continuously differential and  $Y_t = V(t, X_t)$

$$\begin{aligned}
 dY_t &= V_t(t, X_t)dt + V_x(t, X_t)dX_t + \frac{1}{2} V_{xx}(t, X_t)\sigma^2(t, X_t)dt \\
 &= \left( V_t(t, X_t) + V_x(t, X_t)b(t, X_t) + \frac{1}{2} V_{xx}(t, X_t)\sigma^2(t, X_t) \right) dt \\
 &\quad + V_x(t, X_t)\sigma(t, X_t)dW_t
 \end{aligned}$$

# Properties

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# Properties

1 Itô Integral is Martingale

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$$\mathbb{E} \left[ \left( \int_0^t \sigma(s, X_s) dW_s \right)^2 \right] = \mathbb{E} \left[ \int_0^t \sigma(s, X_s)^2 ds \right]$$

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## Corresponding PDE

- In addition,  $V$  solution of

$$V_t + V_x b + \frac{1}{2} V_{xx} \sigma^2 = f$$

$$V(T, \cdot) = U(\cdot)$$

- $Y_t = V(t, X_t) = U(X_T) + \int_t^T f(s, X_s) ds + V_x(t, X_t) \sigma(t, X_t) dW_t$
- Feynman-Kac  $V(t, x) = \mathbb{E} \left[ \int_t^T f(s, X_s) ds + U(X_T) \mid X_t = x \right]$

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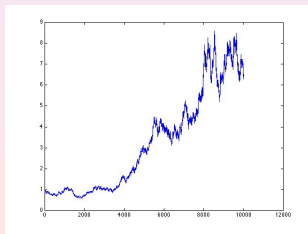
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## B-S Model

- Black, Scholes and Merton(1973)
- $r$  =Interest rate (For simplicity  $r = 0$ ),  $\mu$  = return rate and  $\sigma$  =volatility

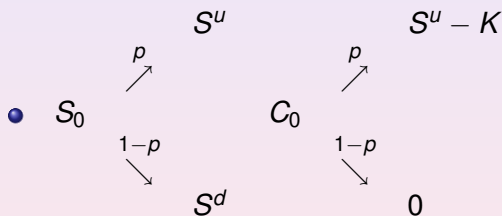
$$dS_t = S_t(\mu dt + \sigma dW_t) \text{ Geometric Brownian Motion}$$

- $S_t = S_0 \exp \left\{ \left( \mu - \frac{1}{2}\sigma^2 \right) t + \sigma W_t \right\} > 0$



## Binomial Tree Model

- $S^d < S_0 < S^u$  and  $S^d < K < S^u$



- Price of a european call option;  $C_0$ ; is a number between  $S^u - K$  and  $0$ ;  $\alpha(S^u - K)$  with  $0 < \alpha < 1$ .
- Using the replicating portfolio:  $\alpha = \frac{S_0 - S^d}{S^u - S^d}$

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## Change of Probability: Binomial Tree

- $\mathbb{Q}$  is a new probability:  
$$C_0 = \mathbb{E}^{\mathbb{Q}} [(S_1 - K)^+] = \alpha(S^u - K) + (1 - \alpha)0$$
- Martingale:  $S_0 = \mathbb{E}^{\mathbb{Q}} [S_1]$
- $S$  is a martingale with respect to  $\mathbb{Q}$ .
- For multiperiod model:  $C_0, C_1, \dots, C_N = (S_T - K)^+$
- $C_i = \mathbb{E}^{\mathbb{Q}} [(S_T - K)^+ | X_1, \dots, X_i] = \mathbb{E}^{\mathbb{Q}} [(S_T - K)^+ | X_i]$

## Change of Probability: Black-Scholes

- $\mathbb{P}$  Probability of  $W_t$
- There exists a probability s.t.  $S_t$  is a martingale under the probability  $\mathbb{Q}$
- $\tilde{W}_t = W_t + \frac{\mu}{\sigma}t$  is a new Brownian motion under  $\mathbb{Q}$
- $dS_t = S_t d\tilde{W}_t$  (Itô integral is a martingale)
- Price of a call option  $C_t = \mathbb{E}^{\mathbb{Q}} [(S^T - K)^+ | W_s; 0 \leq s \leq t]$

# Pricing with PDE Method

- $C_t = V(t, S_t)$
- By Itô formula and the martingale probability:

$$\begin{aligned} V : \mathbb{R}_+ \times \mathbb{R}^d &\rightarrow \mathbb{R} \\ \partial_t V + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} - rV &= 0 \\ V(T, S) &= (S - K)^+ \end{aligned}$$

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## Controlled Diffusion

- Controlled State Process:  
$$dX_t^u = b(t, X_t^u, u_t)dt + \sigma(t, X_t^u, u_t)dW_t$$
- Control:  $u_t$  some Stochastic Process
- Optimization Problem:  $U$  a concave function (Utility)  
For Example:  $U(x) = -e^{-\gamma x}$

$$\sup_u \mathbb{E} \left[ \int_0^T f(s, X_s^u, u_s) ds + U(X_T^u) \right]$$

## Example: Portfolio Management Problem

- Self-Financing Portfolio: Risky Asset  $S_t$  B-S Model + Nonrisky Asset  $S^0 = 1$
- Wealth Process :  $dX_t^\theta = \theta_t (\lambda dt + \sigma dW_t)$   
 ( $\theta :=$  Amount of money invested on risky asset)
- Control Problem: Maximization of the utility of wealth process

$$\sup_{\theta} \mathbb{E} \left[ -\exp \left( -\gamma X_T^\theta \right) \right] = -\exp \left( -\gamma x - \frac{\gamma \lambda^2}{2\sigma^2} (T - t) \right)$$

- Optimal  $\theta$  is  $\hat{\theta} = \frac{\lambda}{\sigma^2}$

## Dynamic Programming Principle

- Value Function

$$V(t, x) := \sup_u \mathbb{E} \left[ \int_t^T f(s, X_s^u, u_s) ds + U(X_T^u) \mid X_t^u = x \right]$$

- Stopping Time  $\tau \geq t$

$$V(t, x) = \sup_{\substack{u_s \\ t \leq s \leq \tau}} \mathbb{E} \left[ \int_t^\tau f(s, X_s^u, u_s) ds + V(\tau, X_\tau^u) \right]$$

## Example: Portfolio Management Problem

- Value Function  $V(t, x) := \sup_u \mathbb{E} [-\exp(-\gamma X_T^\theta) | X_t^u = x]$
- Stopping Time  $\tau \geq t$

$$V(t, x) = \sup_{u_s; t \leq s \leq \tau} \mathbb{E} [V(\tau, X_\tau^u)]$$

# Hamilton-Jacobi-Belman Equations

- $\hat{\text{Ito}}$ :

$$\mathbb{E}[V(\tau, X_\tau^u)] = V(t, x) + \mathbb{E}\left[\int_t^\tau \left(V_t + bV_x + \frac{1}{2}\sigma^2 V_{xx}\right)(s, X_s^u) ds\right]$$

- By DPP,

$$\sup_{u_s; t \leq s \leq \tau} \mathbb{E}\left[\int_t^\tau \left(V_t + bV_x + \frac{1}{2}\sigma^2 V_{xx}\right)(s, X_s^u) ds\right] = 0$$

- $\tau = t + h$ , dividing by  $\frac{1}{h}$  and  $h \rightarrow 0$ :

$$V_t + \sup_u \left\{ bV_x + \frac{1}{2}\sigma^2 V_{xx} \right\} = 0$$

## Example: Portfolio Management Problem

- H-J-B Equation:

$$V_t + \sup_u \left\{ \theta \lambda V_S + \frac{1}{2} \theta^2 \sigma^2 V_{SS} \right\} = 0$$

$$V(T, \cdot) = U(\cdot)$$



$$V_t + \frac{\lambda V_S}{\sigma^2 V_{SS}} = 0$$

$$V(T, \cdot) = U(\cdot)$$

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## Discrete Case, Finite Horizon

- $X_1, \dots, X_N$  State Process
- Control:  $\tau$  Stopping Time
- Optimization Problem:  $U$  a concave function (Utility)

$$\sup_{\tau \text{ Stopping Time}} \mathbb{E} \left[ \sum_{i=1}^{\tau-1} f_i + U(X_\tau) \right]$$

## Backward Induction

- 1  $t = N$  the gain(cost) is  $S_N := U(X_N)$
- 2  $t = N - 1$ , if we stop the gain(cost) is  $U(X_{N-1})$ . If we continue the value of future gain is  $f_{N-1} + \mathbb{E}[S_N|X_{N-1}]$

$$S_{N-1} = \max \{U(X_{N-1}), f_{N-1} + \mathbb{E}[S_N|X_{N-1}]\}$$

- 3  $t = i$ , if we stop the gain(cost) is  $U(X_i)$ . If we continue the value of future gain is  $f_i + \mathbb{E}[S_{i+1}|X_i]$

$$S_i = \max \{U(X_i), f_i + \mathbb{E}[S_{i+1}|X_i]\}$$

- 4  $t = 1$ ,  $S_1 = \sup_{\tau} \mathbb{E} \left[ \sum_{i=1}^{\tau-1} f_i + U(X_{\tau}) \right]$

# Optimal Stopping and Dynamic Programming, Upgrading to Infinite Horizon

- Markovian Case,  $S_i = V(X_i)$

$$V(x) = \sup_{\tau \text{ Stopping Time}} \mathbb{E} \left[ \sum_{i=1}^{\tau-1} f_i + U(X_\tau) \mid X_1 = x \right]$$

- $V(x) \geq U(x)$ . Equality Occures when we it is optimal to stop.
- $\tau^*$  = the first time  $i$  such that  $V(X_i) = U(X_i)$
- Stopping region in the state space;  $\{x \mid V(x) = U(x)\}$
- $V(X_i) = \max \left\{ U(X_i), \mathbb{E} \left[ \sum_{j=i}^{\tau-1} f_j + V(X_\tau) \mid X_i \right] \right\}$

## Continuous Case

- State Process:  $dX_t = b(X_t)dt + \sigma(X_t)dW_t$
- Control:  $\tau$  Stopping Time
- Optimization Problem:  $U$  a concave function (Utility)

$$\sup_{\tau \text{ Stopping Time}} \mathbb{E} \left[ \int_0^{\tau} f(X_s) ds + U(X_{\tau}) \right]$$



# Optimal Stopping and Dynamic Programming

- Value Function

$$V(x) := \sup_{\tau} \mathbb{E} \left[ \int_0^{\tau} f(X_s) ds + U(X_{\tau}) \mid X_0 = x \right]$$

- $V(x) \geq U(x)$ . The equality occurs when it is optimal to stop.
- $\tau^*$  = the first time  $t$  that  $V(X_t) = U(X_t)$
- if  $\tau \leq \tau^*$  then

$$V(x) = \mathbb{E} \left[ \int_0^{\tau} f(X_s) ds + V(X_{\tau}) \mid X_0 = x \right]$$

# Basic Equation and Free Boundary Problem

- if  $\tau \leq \tau^*$  then

$$V(x) = \mathbb{E} \left[ \int_0^\tau f(X_s) ds + V(X_\tau) | X_\tau \right]$$

- By Itô:

$$b(x)V_x + \frac{1}{2}\sigma^2(x)V_{xx} = f(x)$$

- Free Boundary  $V(x) = U(x)$
- Smooth passage from free boundary

## Example: American Put Option

- Risky Asset  $S_t$  B-S Model
- American Put Option: Option to sell risky asset at price  $K$  at any time  $\tau$
- Contingent Claim:  $(K - S_\tau)^+$
- Optimization Problem:

$$V(s) = \sup_{\tau} \mathbb{E} [(K - S_\tau)^+ | S_0 = s] = \text{Not Closed Form}$$

- Optimal  $\tau^*$  is the first time  $t$  that the price of American Put  $V(S_t)$  is equal to  $(K - S_t)^+$

## Example: Dynamic Programming and Basic Equation

- if  $\tau \leq \tau^*$  then

$$V(s) = \mathbb{E}[V(X_\tau) | \mathcal{S}_\tau]$$

- By Itô:

$$\mu s V_s + \frac{1}{2} \sigma^2 s^2 V_{ss} = 0 \text{ Infinite Horizon}$$

- Free Boundary  $V(s) = (K - s)^+$